Bianchi type-I string cosmological model in the presence of a magnetic flux: exact and qualitative solutions

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Abstract

A Bianchi type-I cosmological model in the presence of a magnetic flux along a cosmological string is investigated. The objective of this study is to generate solutions to the Einstein equations using a few tractable assumptions usually accepted in the literature. The analytical solutions are supplemented with numerical and qualitative analysis. In the frame of the present model the evolution of the Universe and other physical aspects are discussed.

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1 Introduction

Since the observed Universe is almost homogeneous and isotropic, the space-time is usually described by a Friedman-Lamaitre-Robertson-Walker (FLRW) cosmology. Although this is a good approximation, the recent observations suggest the necessity of exploring beyond it. Also it is believed that in the early Universe the FLRW model does not give a correct matter description. The anomalies found in the cosmic microwave background (CMB) and the large structure observations stimulated a growing interest in anisotropic cosmological models of the Universe.

Recently, cosmic strings have drawn considerable interest among the researchers for various aspects such as the study of the early Universe. The presence of cosmic strings in the early Universe could be explained using grand unified theories. These strings arise during the phase transition after the Big Bang explosion as the temperature goes down below some critical point. It is believed that the existence of strings in the early Universe gives rise to the density fluctuations which leads to the formation of the galaxies. Also the cosmic strings have been used in attempts to investigate anisotropic dark energy component including a coupling between dark energy and a perfect fluid (dark matter) [1]. The cosmic string is characterized by a stress energy tensor and it is coupled to the gravitational field.

The presence of magnetic fields in galactic and intergalactic spaces is well established and their importance in astrophysics is generally acknowledged (see e. g. the reviews [2, 3] and references therein). In spite of the fact that the present day magnitude of the magnetic energy is very small in comparison with the matter density, it might not have been negligible during early stages of the evolution of the Universe. Any cosmological model which contains magnetic fields is necessarily anisotropic taking into account that the magnetic field vector implies a preferred spatial direction.

Among many possible alternatives, the simplest and most theoretically appealing of anisotropic models are Bianchi type-I (BI). For studying the effects of anisotropies in the early Universe on the present observed Universe, BI models have been investigated from different points of view. In this class of models it is possible to accommodate the presence of cosmic strings as an example of an anisotropy of space-times generated by one dimensional topological defects.

In what follows we shall investigate the evolution of BI cosmological models in presence of a cosmic string and magnetic fluid. The paper has the following structure. We shall review the basic equations of an anisotropic BI model in the presence of a system of cosmic string and magnetic field. In Section 3 we introduce a few plausible assumptions and investigate their outcomes. The objective of this treatment is to generate exact solutions to the Einstein equations, supple-

mented with numerical and qualitative analysis. At the end we shall summarize the results and outline future prospects.

2 Fundamental Equations and general solutions

The line element of a BI Universe is

$$ds^2 = (dt)^2 - a_1(t)^2(dx^1)^2 - a_2(t)^2(dx^2)^2 - a_3(t)^2(dx^3)^2\,. \tag{2.1}$$

There are three scale factors a_i $(i = 1,2,3)$ which are functions of time t only and consequently three expansion rates. In principle all these scale factors could be different and it is useful to express the mean expansion rate in terms of the average Hubble rate:

$$H = \frac{1}{3}\Bigl(\frac{\dot a_1}{a_1} + \frac{\dot a_2}{a_2} + \frac{\dot a_3}{a_3}\Bigr), \tag{2.2}$$

where over-dot means differentiation with respect to t.

In the absence of a cosmological constant, the Einstein's gravitational field equation has the form

$$\frac{\ddot a_2}{a_2} + \frac{\ddot a_3}{a_3} + \frac{\dot a_2}{a_2}\frac{\dot a_3}{a_3} = \kappa T_1^1\,, \tag{2.3a}$$

$$\frac{\ddot a_3}{a_3} + \frac{\ddot a_1}{a_1} + \frac{\dot a_3}{a_3}\frac{\dot a_1}{a_1} = \kappa T_2^2\,, \tag{2.3b}$$

$$\frac{\ddot a_1}{a_1} + \frac{\ddot a_2}{a_2} + \frac{\dot a_1}{a_1}\frac{\dot a_2}{a_2} = \kappa T_3^3\,, \tag{2.3c}$$

$$\frac{\dot a_1}{a_1}\frac{\dot a_2}{a_2} + \frac{\dot a_2}{a_2}\frac{\dot a_3}{a_3} + \frac{\dot a_3}{a_3}\frac{\dot a_1}{a_1} = \kappa T_0^0\,, \tag{2.3d}$$

where κ is the gravitational constant. The energy momentum tensor for a system of cosmic string and magnetic field in a comoving coordinate is given by

$$T_\mu^\nu = \rho u_\mu u^\nu - \lambda x_\mu x^\nu + E_\mu^\nu\,, \tag{2.4}$$

where ρ is the rest energy density of strings with massive particles attached to them and can be expressed as $\rho = \rho_p + \lambda$, where ρ_p is the rest energy density of the particles attached to the strings and λ is the tension density of the system of strings [4, 5, 6] which may be positive or negative. Here u_i is the four velocity and x_i is the direction of the string, obeying the relations

$$u_i u^i = -x_i x^i = 1, \quad u_i x^i = 0\,. \tag{2.5}$$

In (2.4) $E_{\mu\nu}$ is the electromagnetic field given by Lichnerowich [7]. In our case the electromagnetic field tensor $F^{\alpha\beta}$ has only one non-vanishing component, namely

$$F_{23} = h, \tag{2.6}$$

where h is assumed to be constant. For the electromagnetic field E_μ^ν one gets the following non-trivial components

$$E_0^0 = E_1^1 = -E_2^2 = -E_3^3 = \frac{h^2}{2\bar{\mu} a_2^2 a_3^2}. \tag{2.7}$$

where $\bar{\mu}$ is a constant characteristic of the medium and called the magnetic permeability. Typically $\bar{\mu}$ differs from unity only by a few parts in 10^5 ($\bar{\mu} > 1$ for paramagnetic substances and $\bar{\mu} < 1$ for diamagnetic).

Choosing the string along x^1 direction and using comoving coordinates we have the following components of energy momentum tensor [8]:

$$T_0^0 = \rho + \frac{h^2}{2\bar{\mu}}\frac{a_1^2}{\tau^2}, \tag{2.8a}$$

$$T_1^1 = \lambda + \frac{h^2}{2\bar{\mu}}\frac{a_1^2}{\tau^2}, \tag{2.8b}$$

$$T_2^2 = -\frac{h^2}{2\bar{\mu}}\frac{a_1^2}{\tau^2}, \tag{2.8c}$$

$$T_3^3 = -\frac{h^2}{2\bar{\mu}}\frac{a_1^2}{\tau^2}, \tag{2.8d}$$

where we introduce the volume scale of the BI space-time

$$\tau = a_1 a_2 a_3, \tag{2.9}$$

namely, $\tau = \sqrt{-g}$ [9]. It is interesting to note that the evolution in time of τ is connected with the Hubble rate (2.10):

$$\frac{\dot{\tau}}{\tau} = 3H. \tag{2.10}$$

In view of $T_2^2 = T_3^3$ from (2.3b), (2.3c) one finds

$$a_2 = c_3 a_3 \exp\left(d \int \frac{dt}{\tau}\right), \tag{2.11}$$

with c_3, d some real constants.

Since the metric functions can be expressed in terms of τ, let us first derive the equation for τ. Summation of Einstein equations (2.3a), (2.3b), (2.3c) and 3 times (2.3d) gives

$$\frac{\ddot{\tau}}{\tau} = \frac{3}{2}\kappa\Big(\rho + \frac{\lambda}{3} + \frac{h^2}{3\bar{\mu}}\frac{a_1^2}{\tau^2}\Big)\,. \tag{2.12}$$

Taking into account the conservation of the energy-momentum tensor, i.e., $T^{\nu}_{\mu;\nu} = 0$, after a little manipulation of (2.8) one obtains

$$\dot{\rho} + \frac{\dot{\tau}}{\tau}\rho - \frac{\dot{a}_1}{a_1}\lambda = 0\,. \tag{2.13}$$

3 Some examples and explicit solutions

The above equations involve some unknowns and we need some supplementary relations between them to have a tractable problem. It is customary to assume a relation between ρ and λ in accordance with the state equations for strings. The simplest one is a proportionality relation [4]:

$$\rho = \alpha\lambda\,. \tag{3.14}$$

The most usual choices of the constant α are

$$\alpha = \begin{cases} 1 & \text{geometric string} \\ 1+\omega & \omega \geq 0, \quad p \text{ string or Takabayasi string} \\ -1 & \text{Reddy string}\,. \end{cases} \tag{3.15}$$

In order to solve the Einstein equations completely, we need also to impose some additional conditions. In what follows we shall illustrate the general considerations by two examples.

3.1 Case 1: Hubble rate proportional to an eigenvalue of the shear tensor

As a first example, we shall follow the condition introduced by Bali [10] assuming that the average Hubble rate H (2.2) in the model is proportional to the eigenvalue σ_1^1 of the shear tensor σ_μ^ν. We shall only give the briefest account here and for further information the reader should consult [11].

For the BI space-time we have

$$\sigma_1^1 = -\frac{1}{3}\Big(4\frac{\dot a_1}{a_1} + \frac{\dot a_2}{a_2} + \frac{\dot a_3}{a_3}\Big)\,. \tag{3.16}$$

Writing the aforementioned proportionality condition as

$$H = p\sigma_1^1\,, \tag{3.17}$$

one comes to the following relation

$$a_1 = c_{23}\big(a_2 a_3\big)^q\,, \tag{3.18}$$

where $q = -(p+1)/(4p+1)$ is related to the proportionality constant p and c_{23} is an integration constant.

In this case (2.13) takes the form

$$\dot\rho + \big(\rho - \frac{q}{q+1}\lambda\big)\frac{\dot\tau}{\tau} = 0\,, \tag{3.19}$$

and (2.12) now reads

$$\ddot\tau = \frac{3}{2}\kappa(\rho + \frac{\lambda}{3})\tau + b_1 \tau^{(q-1)/(q+1)}\,, \tag{3.20}$$

where $b_1 = \kappa\frac{h^2}{2\mu}c_{23}^{2/(q+1)}$ is another constant.

Let us now study (3.19) and (3.20) for different equations of state. Assuming the relation (3.14) between the pressure of the perfect fluid ρ and the tension density λ from (3.19) one finds

$$\frac{\dot\rho}{\rho} = (\frac{q}{\alpha(q+1)} - 1)\frac{\dot\tau}{\tau}\,, \tag{3.21}$$

with the solution

$$\rho = b_2 \tau^{\frac{q}{\alpha(q+1)} - 1}\,, \tag{3.22}$$

while the equation for τ reads

$$\ddot\tau = \frac{3}{2}\kappa b_2(1 + \frac{1}{3\alpha})\tau^{\frac{q}{\alpha(q+1)}} + b_1\tau^{\frac{q-1}{q+1}}\,, \tag{3.23}$$

involving another constant of integration b_2.

This equation can be set in the following form

$$\dot\tau = \sqrt{\frac{\kappa b_2(3\alpha+1)(q+1)}{q+\alpha(q+1)}\tau^{1+q/\alpha(q+1)} + \frac{b_1(q+1)}{q}\tau^{2q/(q+1)} + b_3} \tag{3.24}$$

where b_3 is an integration constant.

More details for this model characterized by (3.17) including some numerical computations are given in [11].

3.2 Case 2: Constraints on relative shear anisotropy parameters

The second case is more involved taking into account that the anisotropy is connected with the values of the shear distortions. Besides the generalized Hubble parameter (2.2) we consider two relative shear anisotropy parameters defined by [12]:

$$R = \frac{1}{H}\Big(\frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2}\Big), \tag{3.25a}$$

$$S = \frac{1}{H}\Big(\frac{\dot{a}_1}{a_1} - \frac{\dot{a}_3}{a_3}\Big). \tag{3.25b}$$

When $R = S = 0$ the universe will be the isotropic flat Friedman Universe. In this second example we shall assume a deviation from the Friedman model.

We shall consider three different assumptions regarding the relative shear anisotropy parameters. Although these assumptions are distinct, they entail mild time dependencies for R and S. More exactly the time evolution of the scale factors a_i is assumed to be very similar to that of the average Hubble rate H.

3.2.1 Assumption 1: R is constant

Let us assume that $R = r_1$ with r_1 a constant. In view of (2.2) one finds,

$$a_1 = c_2 a_2 \tau^{r_1/3}. \tag{3.26}$$

Then, together with (2.9), (2.11) we find the following expressions for metric functions

$$a_1 = (c_2^2 c_3)^{1/3} \tau^{(3+2r_1)/9} e^{\frac{d}{3}\int \frac{dt}{\tau}}, \tag{3.27a}$$

$$a_2 = (c_2^{-1} c_3)^{1/3} \tau^{(3-r_1)/9} e^{\frac{d}{3}\int \frac{dt}{\tau}}, \tag{3.27b}$$

$$a_3 = (c_2 c_3^2)^{-1/3} \tau^{(3-r_1)/9} e^{\frac{-2d}{3}\int \frac{dt}{\tau}}. \tag{3.27c}$$

In this case (2.13) takes the form

$$\dot{\rho} + \Big(\rho - \frac{3+2r_1}{9}\lambda\Big)\frac{\dot{\tau}}{\tau} - \frac{d}{3\tau}\lambda = 0, \tag{3.28}$$

whereas, for τ in this case we have

$$\frac{\ddot{\tau}}{\tau} = \frac{3}{2}\kappa\Big(\rho + \frac{\lambda}{3} + \frac{h^2}{3\bar{\mu}}(c_2^2 c_3)^{2/3} \tau^{(4r_1-12)/9} e^{\frac{2d}{3}\int \frac{dt}{\tau}}\Big). \tag{3.29}$$

Comparing this equation with the corresponding one (3.23) from **Case 1**, it results that in **Case 2** the equation of evolution for τ is more intricate. In [11] we gave some asymptotic solutions of this equation. Here we give more general solution to the equation in question.

In what follows we express equations (3.28) and (3.29) as a new system of differential equations which is easier to analyze:

$$\dot{\tau} = 3H\tau, \tag{3.30a}$$

$$\dot{T} = \frac{T}{\tau}, \tag{3.30b}$$

$$\dot{H} = -3H^2 + \frac{1}{2}\kappa\Big(\frac{3\alpha+1}{3\alpha}\rho + \frac{h^2}{3\bar{\mu}}(c_2^2 c_3)^{\frac{2}{3}}\tau^{\frac{(4r_1-12)}{9}}T^{\frac{2d}{3}}\Big), \tag{3.30c}$$

$$\dot{\rho} = \Big(\frac{d}{3\alpha}\frac{1}{\tau} - \frac{9\alpha-3-2r_1}{3\alpha}H\Big)\rho\,. \tag{3.30d}$$

This system was investigated qualitatively using numerical methods. To numerically integrate the above system of differential equations with rational polynomials in the right hand sides we use the following method: each of the variables $V = \{\tau, H, \rho, T\}$ was substituted by $V = V_s/V_c$ with $V_s^2 + V_c^2 = 1$. Further we find the common denominator of the right hand sides and moving away it we find the new system of equation with new parameters. This removing is equivalent to the substitution of an independent variable. In the new variable the right hand sides of the differential equations, as well as the integrable functions themselves, possess finite variation. So in the compact region the integral curves happen to be stable if the initial value is given inside the region. It does not change if we numerically integrate them in natural parameters (along the length of the curve).

For exemplification we make the following specific choices: $r_1 = 5.25$, $d = 1.5$, $\kappa = 1$, $\bar{\mu} = 1$, $\mathfrak{J} = 1$, $c_3 = 1$, and $c_2 = 1$. We have investigated the system for two different values of α, namely, $\alpha = 1$ and $\alpha = -1$. To plot the graphs[1], it is more convenient to use instead of the function f, $\arctan f$ in order to convert their infinite ranges of variation into finite ones. The initial condition we give on the line where $H = 0$ and study the evolution of all the four functions both through the past and future. In the Figs. 1 and 2 we plot the evolution of τ for a positive and negative α, respectively, while in Figs. 3 and 4 we do the same for energy density. In Figs. 5 and 6, **3D** graph of τ, H and ρ has been illustrated for a positive and negative α, respectively for the following initial values:

[1]Figures of this article are in color in the electronic version.

initial values						
τ	1.08	3.53	0.50	1.37	6.28	1.78
H	-1	-1	0	0	0	1
ρ	1	1	1	1	1	1
T	.1	.1	.1	.1	.1	.1

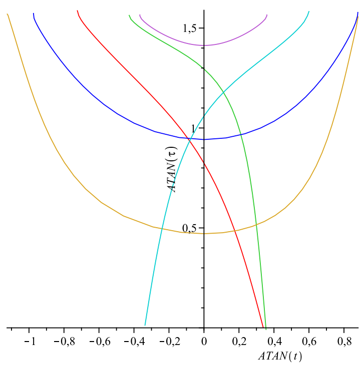

Figure 1: Evolution of the BI universe for a positive α.

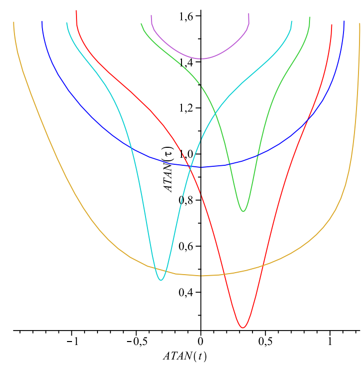

Figure 2: Evolution of the BI universe for a negative α.

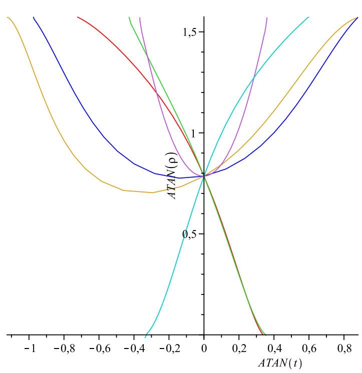

Figure 3: Evolution of the energy density for $\alpha > 0$.

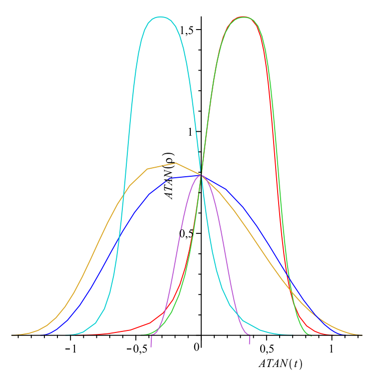

Figure 4: Evolution of the energy density for $\alpha < 0$.

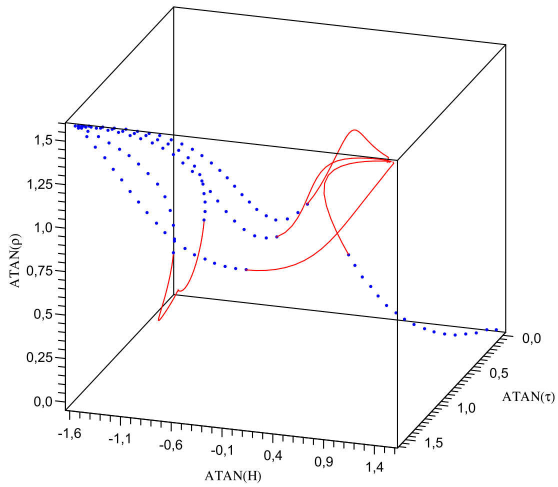

Figure 5: 3D view of $\arctan(\rho)$, $\arctan(H)$ and $\arctan(\tau)$ for $\alpha > 0$.

The system of four time-depended variables τ, H, ρ, T possesses integral curves in 4D space. The graphs of $\rho(t)$ and $\tau(t)$ are the projections from the extended space of dimension 4+1 on the plane with dimension 1+1. Therefore, the projections of integral curves can intersect. In the figures the integral curves correspond to different initial conditions with similar other parameters. In the figures 5 and 6 the dot lines leads to the past, the solid ones to the future, while the point is the one where the initial condition of integration was given.

There is also the possibility to assume that S is constant and the analysis develops in the aforesaid manner, practically interchanging a_2 with a_3.

3.2.2 Assumption 2: R and S differ by a constant

Let us now assume the case when $R - S = r_2$ with r_2 a constant. This assumption leads to

$$a_2 = c_3 a_3 \tau^{r_2/3}, \tag{3.31}$$

which together with (2.11) gives

$$\tau = \tau_0 + (3d/r_2)t\,. \tag{3.32}$$

Thus we find τ to be a linear function of t. It should be noted that analogical results occurs when the BI Universe is filled with stiff fluid. On the other hand from (2.9) one finds

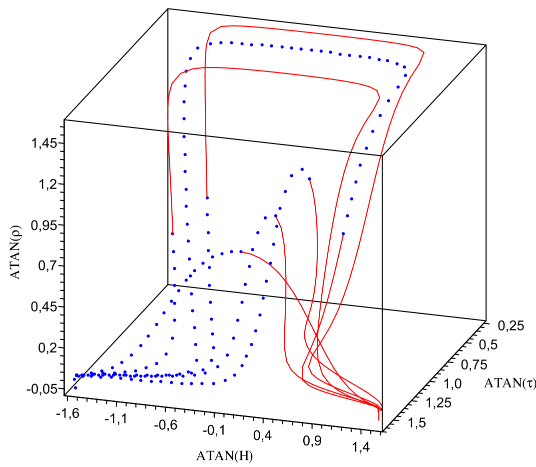

Figure 6: 3D view of $\arctan(\rho)$, $\arctan(H)$ and $\arctan(\tau)$ for $\alpha < 0$.

$$a_1 = (1/c_3) a_3^{-2} \tau^{1-r_2/3}. \tag{3.33}$$

In view of (3.32), (3.33) and (3.14) from (2.12) one finds

$$\rho = -\frac{\alpha h^2}{\bar{\mu} c_3^2 (3\alpha + 1)} \frac{\tau^{(4r_2-6)/3}}{a_3^4}. \tag{3.34}$$

Finally, inserting (3.34), (3.33) into (2.13) we find

$$a_3 = c_4 \tau^{(\alpha(3-2r_2)-3+r_2)/6(2\alpha-1)}, \tag{3.35}$$

with c_4 another constant.

Thus the Einstein system of equations has been completely solved .

3.2.3 Assumption 3: R and S are proportional

Let us now assume the case when $R = r_3 S$ with r_3 being some constant. This leads to the relation

$$\frac{a_1^{1-r_3} a_3^{r_3}}{a_2} = c_{13}, \tag{3.36}$$

with c_{13} being some arbitrary constant. This relation together with (2.11) and (2.9) gives

$$
\begin{aligned}
a_1 &= c_3^{(r_3+1)/3(1-r_3)} c_{13}^{2/3(1-r_3)} \tau^{1/3} e^{\frac{d}{3}\frac{r_3+1}{1-r_3}\int\frac{dt}{\tau}}\,, && (3.37a)\\
a_2 &= c_3^{(1-2r_3)/3(1-r_3)} c_{13}^{-1/3(1-r_3)} \tau^{1/3} e^{\frac{d}{3}\frac{1-2r_3}{1-r_3}\int\frac{dt}{\tau}}\,, && (3.37b)\\
a_3 &= c_3^{(r_3-2)/3(1-r_3)} c_{13}^{-1/3(1-r_3)} \tau^{1/3} e^{\frac{d}{3}\frac{r_3-2}{1-r_3}\int\frac{dt}{\tau}}\,. && (3.37c)
\end{aligned}
$$

The system analogue to (3.30) is:

$$
\begin{aligned}
\dot{\tau} &= 3H\tau\,, && (3.38a)\\
\dot{T} &= \frac{T}{\tau}\,, && (3.38b)\\
\dot{H} &= -3H^2 + \frac{1}{2}\kappa\Big(\frac{3\alpha+1}{3\alpha}\rho + \frac{h^2}{3\bar{\mu}} c_3^{\frac{2}{3}\frac{r_3+1}{1-r_3}} c_{13}^{\frac{4}{3(1-r_3)}} \tau^{4/3} T^{\frac{2d}{3}\frac{1+r_3}{1-r_3}}\Big), && (3.38c)\\
\dot{\rho} &= \Big(\frac{d}{3\alpha}\frac{1+r_3}{1-r_3}\frac{1}{\tau} + \frac{1-3\alpha}{\alpha}H\Big)\rho\,. && (3.38d)
\end{aligned}
$$

This system is quite similar to (3.30) and choosing suitable parameters we find solutions which look as the ones illustrated in the previous graphs.

4 Summary and outlook

We have offered an investigation of an anisotropic cosmological BI model. Having in mind the complexity of the model we used some tractable assumptions regarding the parameters entering the model. For different assumptions regarding the shear anisotropy parameters (3.25a) and (3.25b) we get interesting models deserving the study. The analytical results are supplemented with numerical and qualitative analysis describing the evolution of a BI Universe for different values of the parameters.

In our further studies [13] we should like to see how the model isotropises at late times. Any realistic model must lead to isotropisation necessary for compatibility with standard cosmological models at late times and in agreement with current observations. Also it is important to know how stable is the model to perturbations of the parameters.

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